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THE REALIZATION OF NAMBU — JONA-LASINIO TYPE MODEL ON PHYSICAL FIELDS

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Method of dynamical mapping for the Heisenberg fields onto physical fields in the four fermion interaction Hamiltonian (nonrelativistic variant of NJL model) is used to calculate: energy of physical vacuum, one-particle excitation energy spectrum, wave function and mass of the bound state of two excitations.

Реализация модели Намбу — Йона-Лазинио на физических полях

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Методом динамического отображения гейзенберговских полей на физические поля для гамильтониана с четырехфермионным взаимодействием вычислены: энергия физического вакуума, энергетический спектр одночастичных возбуждений, волновая функция и масса связанного состояния двух возбуждений.

The investigation of the bound-state problem in the frame of quantum field theory may be done by using, at least, two methods. The first one is based on the search for self-consistene solutions of Schwinger — Dyson (SD) equation for the full propagator of interacting particles and Bete — Solpiter (BS) equation for the vertex Green function (see [1] and references there). Another one deals straightly with the state vectors [2,3].

The Nambu and Jona-Lasinio (NJL) model [4] fits from any points of view to study the possibility of producing the bound states. Though originally it was solved by the Green function method, its resemblance to the superconductive type models allows one to use both the first and the second methods.

In the present paper we consider the most simple nonrelativistic variant of NJL model, more closely related to the nonlinear Heisenberg theory [5], to demonstrate the advantages of the second approach. We find the representation for the Heisenberg fields via «physical» fields (dynamical

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mapping [6]) describing the collective degrees of freedom (excitations). These excitations can form bound states, and the momenta of the excitations turn out to be strictly correlated between each other. The exact expression for the wave function of the bound states is written out, but for the energy (mass) of the bound states we obtain nonlinear integral equation, the type of equation for the energy gap. The spectrum of the one-particle excitations and the energy of the ground states have been found as well. As a conclusion we show briefly the correspondence of these two methods.

The physical states, by the definition, are the states upon which a Hamiltonian is diagonal in a weak sense:

$$\langle \mathbf{k} | H | \mathbf{k}' \rangle = \langle \mathbf{k} | \int d^3q B^+(\mathbf{q}) B(\mathbf{q}) E(\mathbf{q}) | \mathbf{k}' \rangle + W_0, \tag{1}$$

where $B(\mathbf{k})|0\rangle = 0$, $|\mathbf{k}\rangle = B^+(\mathbf{k})|0\rangle$ and

$$[H, B^+(\mathbf{k})]|0\rangle = E(\mathbf{k})|0\rangle$$

and $E(\mathbf{k})$- energy spectrum of physical particles.

Consider now the Hamiltonian of our model:

$$H = \int d^3x \left[\psi_\alpha^+(x)\, \varepsilon(\nabla)\, \psi_\alpha(x) + \frac{\lambda}{4} \chi^+(x) \chi(x) \right], \tag{2}$$

where α is spin index running over 1,2,$\varepsilon(\nabla)$ is energy spectrum of free fermions, defined by the condition

$$\varepsilon(\nabla)\, e^{i\mathbf{k}\mathbf{x}} = \varepsilon(\mathbf{k})\, e^{i\mathbf{k}\mathbf{x}}, \tag{3}$$

and*

$$\chi(x) = \varepsilon_{\alpha\beta} \psi_\alpha(x) \psi_\beta(x), \quad \chi^+(x) = \varepsilon_{\alpha\beta} \psi_\beta^+(x) \psi_\alpha^+(x).$$

Let us formulate the problems we want to solve demonstrating the efficiency of the physical field representation.

- i) Connection of the Heisenberg fields ψ with physical ones φ, i.e., dynamical mapping ψ on φ.
- ii) Stability of vacuum and its energy.
- iii) Spectrum of one particle physical state $E(\mathbf{k})$.
- iv) Spectrum and wave function of two-particle state.
- v) Correspondence with the usual approach.

We will give the solutions of the outlined problems following the list.

i) In our case the dynamical mapping has the form:

$$\psi_\alpha(x) = u_0 \phi_\alpha(x) + v_0 e^{i\alpha(x)}\, \varepsilon_{\alpha\beta} \phi_\beta^+(x),$$

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*$\varepsilon_{12} = -\varepsilon_{21}$, $\varepsilon_{\alpha\beta}\varepsilon_{\alpha\gamma} = \delta_{\beta\gamma}$

$$\psi_{\alpha}^{+}(x) = u_0\phi_{\alpha}^{+}(x) + v_0 e^{-i\alpha(x)}\,\varepsilon_{\alpha\beta}\phi_{\beta}(x), \tag{4}$$

where $u_0^2 + v_0^2 = 1$, with $\phi_{\alpha}(x)$ defined as

$$\phi_{\alpha}(x) = \frac{1}{(2\pi)^{3/2}} \int d^3k g(\mathbf{k}) e^{i\mathbf{k}\mathbf{x} - iE(\mathbf{k})t} A_{\alpha}(\mathbf{k}), \tag{5}$$

$$\left\{A_{\alpha}(\mathbf{k}), A_{\beta}^{+}(\mathbf{q})\right\} = \delta_{\alpha\beta}\delta(\mathbf{k} - \mathbf{q}), \tag{6}$$

$E(\mathbf{k})$ is unknown yet excitation spectrum of $A_{\alpha}^{+}(\mathbf{k})$. The physical meaning of $g(\mathbf{k})$ is clear enough, it is one-particle wave function and it cannot be calculated in the frames of this model (free parameter).

ii) The vacuum is now defined with respect to the physical fields $A_{\alpha}^{+}(\mathbf{k})$:

$$A_{\alpha}(\mathbf{k})|0\rangle = 0. \tag{7}$$

Consider the action of the Hamiltonian (2) on this vacuum. We have:

$$H|0\rangle = \mathrm{const}_0|0\rangle + \Delta H(2)|0\rangle. \tag{8}$$

$\Delta H(2)$ being expressed in terms of the creation operators $A_{\alpha}^{+}(\mathbf{k})$ has the form

$$\Delta H(2) = u_0 v_0 \varepsilon_{\alpha\beta} \int d^3x \phi_{\alpha}^{+}(x) \left(\varepsilon(\;) + \frac{\lambda}{2v^*}\right) e^{i\mathbf{k}_0\mathbf{x}} \phi_{\beta}^{+}(x) =$$
$$= \int d^3k D(\mathbf{k}) \varepsilon_{\alpha\beta} A_{\alpha}^{+}\left(\frac{\mathbf{k}_0}{2} - \mathbf{k}\right) A_{\beta}^{+}\left(\frac{\mathbf{k}_0}{2} + \mathbf{k}\right), \tag{9}$$

where

$$D(\mathbf{k}) = u_0 v_0 g\left(\mathbf{k} - \frac{\mathbf{k}_0}{2}\right) \bar{g}\left(\mathbf{k} + \frac{\mathbf{k}_0}{2}\right) \left(\varepsilon\left(\mathbf{k} - \frac{\mathbf{k}_0}{2}\right) + \frac{\lambda}{2V^*}\right)$$

$$\frac{1}{V^*} = \frac{1}{(2\pi)^3} \int d^3k |g(\mathbf{k})|^2.$$

Thus, as it follows from (8), the nonexcited state $|0\rangle$ is not an eigenstate of the Hamilonian (2), and the $\Delta H(2)$ term is the source of the nonstationarity of $|0\rangle$. This term describes correlated fermion couple of excitations, moving with the momentum $\mathbf{k}_0$. The physical meaning of the relation (8) is that it points to the existence of energy exchange between the couple and the system of fermions. In order to account this exchange one has to input the term describing the couple into the initial Hamiltonian. So, in the relation

(8) we transfer $\Delta H(2)$ to the left part, thus, redefining the Hamiltonian and taking as a physical Hamiltonian the quantity H_p equal to

$$H_p = :H - \Delta H(2):, \tag{10}$$

where the normal ordering is referred to the Heisenberg fields. If $\Delta H(2) =: \Delta H(2): + \text{const}$, then

$$H_p|0\rangle = H|0\rangle - \Delta H(2)|0\rangle + \text{const}|0\rangle,$$

$$H_p|0\rangle = W_0|0\rangle, \; W_0 = \text{const}_0 + \text{const}, \tag{11}$$

$$\text{const}_0 = 2v_0^2 \frac{V}{V^*}\left(\varepsilon(\mathbf{k}_0) + \frac{\langle \mathbf{k}^2\rangle}{2m} + \frac{\lambda}{2V^*}\right),$$

$$\text{const.} = -\frac{4u_0^4v_0^2}{V^*}\int d^2x\, e^{-i\alpha(x)}\varepsilon(\nabla)e^{i\alpha(x)} - 4\frac{V}{V^*}u_0^2v_0^2\left(\frac{\langle \mathbf{k}^2\rangle}{2m} + \frac{\lambda}{2V^*}\right). \tag{12}$$

Here V is space volume, and

$$\langle \mathbf{k}^2\rangle = \frac{\int d^3k\mathbf{k}^2|g(\mathbf{k})|^2}{\int d^3k|g(\mathbf{k})|^2}. \tag{13}$$

Consequently, after substitution of the counterterm $\Delta H(2)$ into the Hamiltonian (2) the non-excited state $|0\rangle$ becomes stationary, with the energy W_0.

iii) It is easy to see from (1) that this redefinition of the Hamiltonian does not change the one-particle excitation spectrum $E(\mathbf{k})$, the value of which is readily derived using eqs. (1), (2), (4)—(6)

$$E(\mathbf{k}) = |g(\mathbf{k})|^2\left(u_0^2\varepsilon(\mathbf{k}) - v_0^2\varepsilon(\mathbf{k}-\mathbf{k}_0) - \lambda\frac{v_0^2}{V^*}\right). \tag{14}$$

iv) Let us now find how the Hamiltonian H_p acts on a state composed of two excitations with momenta $\mathbf{k}$ and $\mathbf{q}$

$$H_pA_\alpha^+(\mathbf{k})A_\beta^+(\mathbf{q})|0\rangle = (W_0 + E(\mathbf{k}) + E(\mathbf{q}))A_\alpha^+(\mathbf{k})A_\beta^+(\mathbf{q})|0\rangle +$$
$$+ \lambda\frac{g(\mathbf{k})g(\mathbf{q})}{(2\pi)^3}\int d^3xe^{i(\mathbf{k}+\mathbf{q})\mathbf{x}}\phi_\alpha^+(x)\phi_\beta^+(x)0\rangle. \tag{15}$$

If the excitations do not interact with each other, the last term in the sum (15) should vanish. Then, this two-particle excitation will be an eigenstate of the Hamiltonian H_p with the energy being equal to the sum of both excitation energies. That is the case for two excitations with a total spin 1 (symmetrical over $\alpha\beta$). Putting $\mathbf{k} = -\mathbf{q}$ in equation (15) we have:

$$H_p\hat{A}(\mathbf{k})|0\rangle = (W_0 + E(\mathbf{k}) + E(-\mathbf{k}))\hat{A}(\mathbf{k})|0\rangle + \\ + \lambda\frac{|g(\mathbf{k})|^2}{(2\pi)^3}\int d^3q|g(\mathbf{q})|^2\hat{A}^+(\mathbf{q})|0\rangle,$$

$$\hat{A}^+(\mathbf{k}) = \varepsilon_{\alpha\beta}A_\alpha^+(\mathbf{k})A_\beta^+(-\mathbf{k}). \tag{16}$$

Taking the wave packet $\hat{A}^+$:

$$\hat{A}^+ = \int d^3kG(\mathbf{k})\varepsilon_{\alpha\beta}A_\alpha^+(\mathbf{k})A_\beta^+(-\mathbf{k}) \tag{17}$$

and demanding for it to be an eigenstate of the Hamiltonian

$$H_p\hat{A}^+|0\rangle = (W_0 + \mu)\hat{A}^+|0\rangle \tag{18}$$

we come to the equation on the wave function $G(\mathbf{k})$ and the energy of this state μ:

$$\int d^3k\left[G(\mathbf{k})\ (E(\mathbf{k}) + E(-\mathbf{k}) - \mu) + \gamma_0|g(\mathbf{k})|^2\right]\hat{A}^+(\mathbf{k})|0\rangle = 0, \tag{19}$$

where

$$\gamma_0 = \frac{\lambda}{(2\pi)^3}\int d^3q|g(\mathbf{q})|^2G(\mathbf{q}). \tag{20}$$

Hence follows the only solution for $G(\mathbf{k})$:

$$G(\mathbf{k}) = \frac{\gamma_0|g(\mathbf{k})|^2}{\mu - E(\mathbf{k}) - E(-\mathbf{k})}. \tag{21}$$

Substitution of this solution into the relation (20) leads to the equation for μ:

$$1 = \frac{\lambda}{(2\pi)^3}\int d^3k\frac{|g(\mathbf{k})|^4}{\mu - E(\mathbf{k}) - E(-\mathbf{k})}. \tag{22}$$

Let us rewrite (22) in the form

$$1 = -\frac{\lambda M}{(2\pi)^3}\int d^3k\frac{|g(\mathbf{k})|^2}{\mathbf{k}^2 + \delta^2}. \tag{23}$$

This equation determines δ^2 as a function of λ. We will seek solution of (23) in the form of asymptotic series

$$\delta^2 = a_0 + a\lambda + \frac{c_1}{\lambda} + \frac{c_2}{\lambda^2} + \dots, \tag{24}$$

$$\mathbf{k}^2 + \delta^2 = \lambda\left(a + \frac{\mathbf{k}^2 a_0}{\lambda} + \frac{c_1}{\lambda^2} + \frac{c_2}{\lambda^3} + \dots\right).$$

Using the series conversion formulae we can derive

$$\frac{\lambda}{\mathbf{k}^2 + \delta^2} = \frac{1}{a + b\varepsilon + c\varepsilon^2 + \dots} = \frac{1}{a}\left[1 - \frac{b}{a}\varepsilon + \left(\frac{b^2}{a^2} - \frac{c}{a}\right)\varepsilon^2 + \dots\right], \tag{25}$$

where $\varepsilon = 1/\lambda$, $b = a_0 + \mathbf{k}^2$, $c = c_1$, ect. Substituting the series expansion (25) into the equation (23), collecting the terms of the same power over λ and equating them to zero, we will obtain the chain of relations defining the coefficients a_0, a, c_1, ect. Straightforward calculations give the following result:

$$a_0 = -\langle \mathbf{k}^2 \rangle,\ a = -\frac{M}{V^*},$$

$$c_1 = -\frac{m}{V^*}\frac{\int d^3k\,|g(\mathbf{k})|^2(\mathbf{k}^2 - \langle \mathbf{k}^2 \rangle)^2}{\int d^3k\,|g(\mathbf{k})|^2} \equiv -\frac{M}{V^*}\sigma \tag{26}$$

from which we receive the asymptotic series for δ^2 and μ:

$$\delta^2 = -\langle \mathbf{k}^2 \rangle - \frac{M}{V^*}\lambda - \frac{V^*}{M}\cdot\frac{\sigma}{\lambda} + \dots,$$

$$\mu = -\frac{\delta^2}{M} + 2E(0),\ \ 2E(0) = \left(\frac{\mathbf{k}_0^2}{2m} + \frac{\lambda}{V^*}\right)\cdot\left(\frac{m}{M} - 1\right). \tag{27}$$

The magnitude of M is determined from the vacuum energy W_0 minimization condition over the rotation parameters u_0, v_0 and $\alpha(x)$.

From the relations (26) and (27) it is easy to see that the non-perturbative and singular with respect to the coupling constant λ contributions into the energy are defined by the dispersion σ over momentum distribution $|g(\mathbf{k})|^2$ inside the excitation.

In order to show the correspondence with the Green functions' method we will just write out the Shwinger — Dyson and Bete — Solpiter type equations. The word «type» means that we deal with the physical fields, and to pass to the standard SD and BS equations one has to substitute the Heisenberg fields instead of the physical ones using the inverse dynamical mapping. It is not necessary for our aim to do that, so we leave it for reader to do.

The Green functions for the physical fields ϕ are, by the definition, the vacuum expectation values of the T products of them. For the total two point Green function we have:

$$G^P_{\alpha\beta}(\mathrm{x}, t;0) = \langle 0 | T\left(\phi_\alpha(x)\, \phi^+_\beta(0)\right) | 0 \rangle, \tag{28}$$

here we have put the second argument to zero. Its manifest form can be calculated directly, using (5), from which it is easy to find the equation:

$$\left(\frac{\partial}{\partial t} + iE(\nabla)\right) G^P_{\alpha\beta}(\mathrm{x}, t;0) = \delta_{\alpha\beta}\Delta^{(4)}(\mathrm{x}, t). \tag{29}$$

The equation for the vertex Green function follows from the Heisenberg equation on $\phi_\alpha(x)$ using eq.(29):

$$E(\nabla)G^P_{\alpha\beta}(\mathrm{x},t;0) = \langle 0 | T\left(\left[\phi_\alpha(\mathrm{x},t),\mathrm{H}\right], \phi^+_\beta(0)|0\rangle\right). \tag{30}$$

As has been said above, the usual SD and BS equations follow from (29) and (30) under the transition $\phi \to \psi$.

For the conclusion we would like to point to that the physical fields representation method can be generalized for the relativistic case, and next paper will be devoted to this generalization.

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